New generalized Poisson structures

1996 J. Phys. A: Math. Gen. 29 L151

(http://iopscience.iop.org/0305-4470/29/7/001)

J. Phys. A: Math. Gen. **29** (1996) L151–L157. Printed in the UK

LETTER TO THE EDITOR

New generalized Poisson structures

J A de Azcárraga†, A M Perelomov‡ and J C Pérez Bueno§

Departamento de Física Teórica and IFIC, Centro Mixto Universidad de Valencia–CSIC, 46100-Burjassot, Valencia, Spain

Received 11 January 1996

Abstract. New generalized Poisson structures are introduced by using suitable skew-symmetric contravariant tensors of even order. The corresponding 'Jacobi identities' are provided by conditions on these tensors, which may be understood as cocycle conditions. As an example, we provide the linear generalized Poisson structures which can be constructed on the dual spaces of simple Lie algebras.

About twenty years ago, Nambu [1] proposed a generalization of the standard classical Hamiltonian mechanics based on a three-dimensional 'phase space' spanned by a canonical triplet of dynamical variables and on two 'Hamiltonians'. His approach was later discussed by Bayen and Flato [2] and in [3, 4]. The subject laid dormant until recently when a higher order extension of Nambu's approach, involving $(n-1)$ Hamiltonians, was proposed by Takhtajan [5] (see [6] for applications).

Another subject closely related to Hamiltonian dynamics is the study of Poisson structures (PS) (see [7–9]) on a (Poisson) manifold M. A particular case of Poisson structures is that arising when they are defined on the duals of Lie algebras. The class of linear Poisson structures was considered by Lie himself [10, 11], and has been further investigated recently [12–14]. In general, the property which guarantees the Jacobi identity for the Poisson brackets (PB) of functions on a Poisson manifold may be expressed [7, 15] as $[\Lambda, \Lambda] = 0$ where Λ is the bivector field which may be used to define the Poisson structure and $[\ ,\ ]$ is the Schouten–Nijenhuis bracket (SNB) [16, 17]. In the generalizations of Hamiltonian mechanics the Jacobi identity is replaced by a more complicated one (the 'fundamental identity' in [5]).

The aim of this letter is to introduce a new generalization of the standard PS. This will be achieved by replacing the skew-symmetric bivector Λ defining the standard structure by appropriate even-dimensional skew-symmetric contravariant tensor fields $\Lambda^{(2p)}$, and by replacing the Jacobi identity by the condition which follows from $[\Lambda^{(2p)}, \Lambda^{(2p)}] = 0$. In fact, the vanishing of the SNB of $\Lambda^{(2p)}$ with itself allows us to introduce a generalization of the Jacobi identity in a rather geometrical way, and provides us with a clue in the search for a generalized PS. As a result, we differ from other approaches [1, 5]: all our generalized Poisson brackets (GPB) involve an *even* number of functions, whereas this number is arbitrary

† E-mail address: azcarrag@evalvx.ific.uv.es

‡ On leave of absence from: Institute for Theoretical and Experimental Physics, 117259 Moscow, Russia (current e-mail address: perelomo@evalvx.ific.uv.es).

§ E-mail address: pbueno@lie.ific.uv.es

0305-4470/96/070151+07$19.50

(three in [1]) in earlier extensions. Since the most important question once a new Poisson structure is introduced is to present specific examples of it (in other words, solutions of the generalized Jacobi identities which must be satisfied), we shall exhibit, by generalizing the standard linear structure on the dual space $\mathcal{G}^*$ to a Lie algebra $\mathcal{G}$, the linear Poisson structures which may be defined on the duals of all simple Lie algebras. The solution to this problem has, in fact, a cohomological component: the different tensors $\Lambda^{(2p)}$ which can be introduced are related to Lie algebra cohomology cocycles. We shall also discuss the 'dynamics' associated with the GPB here, but will leave a more detailed account of our theory and its cohomological background to a forthcoming publication [18].

Let us recall some facts concerning standard Poisson structures. Let M be a manifold and $\mathcal{F}(M)$ be the associative algebra of smooth functions on M.

Definition 1 (PB). A *Poisson bracket* $\{\cdot,\cdot\}$ on $\mathcal{F}(M)$ is an operation assigning to every pair of functions $f_1, f_2 \in \mathcal{F}(M)$ a new function $\{f_1, f_2\} \in \mathcal{F}(M)$, which is linear in f_1 and f_2 and satisfies the following conditions:
(a) skew-symmetry

$$\{f_1, f_2\} = -\{f_2, f_1\} \tag{1}$$

(b) the Leibniz rule (derivation property)

$$\{f, gh\} = g\{f, h\} + \{f, g\}h \tag{2}$$

(c) the Jacobi identity

$$\tfrac{1}{2}\,\mathrm{Alt}\{f_1, \{f_2, f_3\}\} \equiv \{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0\,. \tag{3}$$

The identities (1), (3) are simply the axioms of a Lie algebra; thus the space $\mathcal{F}(M)$ endowed with the PB $\{\cdot,\cdot\}$ becomes an (infinite-dimensional) Lie algebra, and M is a *Poisson manifold*.

Let x^j be local coordinates on $U \subset M$ and consider PB of the form

$$\{f(x), g(x)\} = \omega^{jk}(x)\partial_j f \partial_k g \qquad \partial_j = \frac{\partial}{\partial x^j} \qquad j, k = 1, \ldots, \dim M\,. \tag{4}$$

Since Leibniz's rule is automatically fulfilled, $\omega^{ij}(x)$ defines a PB if $\omega^{ij}(x) = -\omega^{ji}(x)$ (equation (1)) and equation (3) is satisfied, i.e. if

$$\omega^{jk}\partial_k\omega^{lm} + \omega^{lk}\partial_k\omega^{mj} + \omega^{mk}\partial_k\omega^{jl} = 0\,. \tag{5}$$

The requirements (1) and (2) imply that the PB may be given in terms of a skew-symmetric biderivative, i.e. by a skew-symmetric bivector field ('Poisson bivector') $\Lambda \in \wedge^2(M)$. Locally

$$\Lambda = \tfrac{1}{2}\omega^{jk}\partial_j \wedge \partial_k\,. \tag{6}$$

The condition (5) may be expressed in terms of Λ as $[\Lambda, \Lambda] = 0$ [7, 15]. A skew-symmetric tensor field $\Lambda \in \wedge^2(M)$ such that $[\Lambda, \Lambda] = 0$ defines a *Poisson structure* on M and M becomes a *Poisson manifold*. The PB is then defined by

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) \qquad f, g \in \mathcal{F}(M)\,. \tag{7}$$

Two PS Λ_1, Λ_2 on M are *compatible* if any linear combination of them is again a PS. In terms of the SNB this means that $[\Lambda_1, \Lambda_2] = 0$.

Given a function H, the vector field $X_H = i_{\mathrm{d}H}\Lambda$ (where $i_\alpha\Lambda(\beta) := \Lambda(\alpha, \beta)$, α, β 1-forms), is called a *Hamiltonian vector field* of H. From the Jacobi identity (3) it easily follows that

$$[X_f, X_H] = X_{\{f,H\}}\,. \tag{8}$$

Thus, the Hamiltonian vector fields form a Lie subalgebra of the Lie algebra $\mathcal{X}(M)$ of all smooth vector fields on M. In local coordinates

$$X_H = \omega^{jk}(x)\partial_j H \partial_k \qquad X_H.f = \{H, f\}\,. \tag{9}$$

We recall that the tensor $\omega^{jk}(x)$ appearing in (4), (6) does not need to be non-degenerate; in particular, the dimension of a Poisson manifold M may be odd. Only when Λ has constant rank $2q$ (is *regular*) and the codimension $(\dim M - 2q)$ of the manifold is zero does Λ defines a *symplectic structure*.

We now turn to linear Poisson structures. A real finite-dimensional Lie algebra $\mathcal{G}$ with Lie bracket $[\,.,.]$ defines in a natural way a PB $\{\,.,.\}_{\mathcal{G}}$ on the dual space $\mathcal{G}^*$ of $\mathcal{G}$. The natural identification $\mathcal{G} \cong (\mathcal{G}^*)^*$, allows us to think of $\mathcal{G}$ as a subset of the ring of smooth functions $\mathcal{F}(\mathcal{G}^*)$. Choosing a linear basis $\{\,e_i\,\}_{i=1}^r$ of $\mathcal{G}$, and identifying its components with linear coordinate functions x_i on the dual space $\mathcal{G}^*$ by means of $x_i(x) = \langle x, e_i\rangle$ for all $x \in \mathcal{G}^*$, the fundamental PB on $\mathcal{G}^*$ may be defined by

$$\{x_i, x_j\}_{\mathcal{G}} = C_{ij}^k x_k \qquad i, j, k = 1, \ldots, r = \dim \mathcal{G} \tag{10}$$

using the fact that $[e_i, e_j] = C_{ij}^k e_k$, where C_{ij}^k are the structure constants of $\mathcal{G}$. Intrinsically, the PB $\{.,.\}_{\mathcal{G}}$ on $\mathcal{F}(\mathcal{G}^*)$ is defined by

$$\{f, g\}_{\mathcal{G}}(x) = \langle x, [\,df(x), \mathrm{d}g(x)]\rangle \qquad f, g \in \mathcal{F}(\mathcal{G}^*) \quad x \in \mathcal{G}^*\,. \tag{11}$$

Locally, $[\mathrm{d}f(x), \mathrm{d}g(x)] = e_k C_{ij}^k(\partial f/\partial x_i)(\partial g/\partial x_j)$, $\{f, g\}_{\mathcal{G}}(x) = x_k C_{ij}^k(\partial f/\partial x_i)(\partial g/\partial x_j)$. The above PB $\{.,.\}_{\mathcal{G}}$ is commonly called a *Lie–Poisson bracket*. It is associated with the bivector field $\Lambda_{\mathcal{G}}$ on $\mathcal{G}^*$ locally written as

$$\Lambda_{\mathcal{G}} = C_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \equiv \omega_{ij}\partial^i \wedge \partial^j \tag{12}$$

(cf equation (6)), so that (cf equation (7)) $\Lambda_{\mathcal{G}}(\mathrm{d}f \wedge \mathrm{d}g) = \{f, g\}_{\mathcal{G}}$. It is convenient to note here that $[\Lambda_{\mathcal{G}}, \Lambda_{\mathcal{G}}]_S = 0$ (cf equation (5)) is just the Jacobi identity for $\mathcal{G}$, which may be written as

$$\tfrac{1}{2}\,\mathrm{Alt}(C_{i_1 i_2}^{\rho} C_{\rho i_3}^{\sigma}) \equiv \tfrac{1}{2}\epsilon_{i_1 i_2 i_3}^{j_1 j_2 j_3} C_{j_1 j_2}^{\rho} C_{\rho j_3}^{\sigma} = 0\,. \tag{13}$$

Let β be a closed 1-form on $\mathcal{G}^*$. The associated vector field

$$X_\beta = i_\beta \Lambda_{\mathcal{G}} \tag{14}$$

is an infinitesimal automorphism of $\Lambda_{\mathcal{G}}$, i.e.

$$L_{X_\beta}\Lambda_{\mathcal{G}} = 0 \tag{15}$$

and $[X_f, X_g] = X_{\{f,g\}}$ (equation (8)); this is proved easily using that $L_{X_f} g = \{f, g\}$ and $L_{X_f}\Lambda_{\mathcal{G}} = 0$. It follows from (12) that the Hamiltonian vector fields $X_i = i_{\mathrm{d}x_i}\Lambda_{\mathcal{G}}$ corresponding to the linear coordinate functions x_i, have the expression (cf equation (9))

$$X_i = C_{ij}^k x_k \frac{\partial}{\partial x_j} \qquad i = 1, \ldots, \dim \mathcal{G} \tag{16}$$

so that the Poisson bivector can be written as

$$\Lambda_{\mathcal{G}} = X_i \wedge \frac{\partial}{\partial x_i}\,. \tag{17}$$

Note that this way of writing $\Lambda_{\mathcal{G}}$ is of course not unique. Using the adjoint representation of $\mathcal{G}$, $(C_i)_{\cdot j}^k = C_{ij}^k$ the Poisson bivector $\Lambda_{\mathcal{G}}$ may be rewritten as

$$\Lambda_{\mathcal{G}} = X_{C_i} \wedge \frac{\partial}{\partial x_i} \qquad \left(X_{C_i} = x_k (C_i)_{\cdot j}^k \frac{\partial}{\partial x_j}\right) \tag{18}$$

the vector fields X_{C_i} provide a realization of ad $\mathcal{G}$ in terms of vector fields on $\mathcal{G}^*$.

We now turn to generalized Poisson structures. A rather stringent condition needed to define a PS on a manifold is the Jacobi identity (3). In terms of Λ, this condition is given in a convenient geometrical way by the vanishing of the SNB of $\Lambda \equiv \Lambda^{(2)}$ with itself, $[\Lambda^{(2)}, \Lambda^{(2)}] = 0$. So, it seems natural to consider generalizations of the standard PS in terms of $2p$-ary operations determined by skew-symmetric $2p$-vector fields $\Lambda^{(2p)}$, the case $p = 1$ being the standard one. Since the SNB of two skew-symmetric contravariant tensor fields A, B of degree† a, b satisfies $[A, B] = -(-1)^{ab}[B, A]$, only $[\Lambda', \Lambda'] = 0$ for Λ' of odd degree will be meaningful, since this SNB will vanish identically if Λ' is of even degree.

Bearing this in mind, let us introduce first the GPB.

Definition 2. A generalized Poisson bracket $\{\cdot, \cdot, \ldots, \cdot, \cdot\}$ on M is a mapping $\mathcal{F}(M) \times \overset{2p}{\ldots} \times \mathcal{F}(M) \rightarrow \mathcal{F}(M)$ assigning a function $\{f_1, f_2, \ldots, f_{2p}\}$ to every set $f_1, \ldots, f_{2p} \in \mathcal{F}(M)$ which is linear in all arguments and satisfies the following conditions:
(a) complete skew-symmetry in f_j;
(b) the Leibniz rule: $\forall f_i, g, h \in \mathcal{F}(M)$

$$\{f_1, f_2, \ldots, f_{2p-1}, gh\} = g\{f_1, f_2, \ldots, f_{2p-1}, h\} + \{f_1, f_2, \ldots, f_{2p-1}, g\}h \tag{19}$$

(c) the generalized Jacobi identity: $\forall f_i \in \mathcal{F}(M)$

$$\mathrm{Alt}\{f_1, f_2, \ldots, f_{2p-1}\{f_{2p}, \ldots, f_{4p-1}\}\} = 0\,. \tag{20}$$

Conditions (a) and (b) imply that our GPB is given by a skew-symmetric multiderivative, i.e. by an completely skew-symmetric $2p$-vector field $\Lambda^{(2p)} \in \wedge^{2p}(M)$. Condition (20) will be called the *generalized Jacobi identity*; for $p = 2$ it contains 35 terms (C^{2p-1}_{4p-1} in the general case). It may be rewritten as $[\Lambda^{(2p)}, \Lambda^{(2p)}] = 0$; $\Lambda^{(2p)}$ defines a GPB. Clearly, the above relations reproduce the ordinary case (1)–(3) for $p = 1$. The compatibility condition of the standard case may now be extended in the following sense: two generalized Poisson structures $\Lambda^{(2p)}$ and $\Lambda^{(2q)}$ on M are called *compatible* if they 'commute', i.e. $[\Lambda^{(2p)}, \Lambda^{(2q)}] = 0$. Let us emphasize that this generalized Poisson structure is different from the Nambu structure [1] recently generalized in [5]. Moreover, we shall see later that our generalized linear PS are automatically obtained from *constant* skew-symmetric tensors of order $2p + 1$.

Let x^j be local coordinates on $U \subset M$. Then the GPB has the form

$$\{f_1(x), f_2(x), \ldots, f_{2p}(x)\} = \omega_{j_1 j_2 \ldots j_{2p}} \partial^{j_1} f_1\, \partial^{j_2} f_2 \,\ldots\, \partial^{j_{2p}} f_{2p}\ . \tag{21}$$

where $\omega_{j_1 j_2 \ldots j_{2p}}$ are the coordinates of a completely skew-symmetric tensor which satisfies

$$\mathrm{Alt}(\omega_{j_1 j_2 \ldots j_{2p-1} k}\, \partial^k\, \omega_{j_{2p} \ldots j_{4p-1}}) = 0 \tag{22}$$

as a result of (20). In terms of a skew-symmetric tensor field of order $2p$ the generalized Poisson structure is defined by

$$\Lambda^{(2p)} = \frac{1}{(2p)!}\, \omega_{j_1 \ldots j_{2p}}\, \partial^{j_1} \wedge \ldots \wedge \partial^{j_{2p}}\,. \tag{23}$$

It is then easy to check that the vanishing of the SNB $[\Lambda^{(2p)}, \Lambda^{(2p)}] = 0$ reproduces (22).

† Note that the algebra of multivector fields is a graded superalgebra and that the *degree* of a multivector A is equal to (*order* $A - 1$). Thus, the standard PS defined by Λ is of even order (two) but of odd degree (one).

Let us now define the dynamical system associated with the above generalized Poisson structure. Namely, let us fix a set of $(2p-1)$ 'Hamiltonian' functions $H_1, H_2, \ldots, H_{2p-1}$ and consider the system

$$\dot{x}_j = \{H_1, \ldots, H_{2p-1}, x_j\}$$

or, in general,

$$\dot{f} = \{H_1, \ldots, H_{2p-1}, f\}\,. \tag{24}$$

Definition 3. A function $f \in \mathcal{F}(M)$ is a constant of motion if (24) is zero.

Due to the skew-symmetry, the 'Hamiltonian' functions $H_1, \ldots, H_{2p-1}$ are all constants of motion but the system may have additional ones $h_{2p}, \ldots, h_k;\ k \geqslant 2p$.

Definition 4. A set of functions $(f_1, \ldots, f_k)\,, k \geqslant 2p$ is in *involution* if the GPB vanishes for any subset of $2p$ functions.

Let us also note the following generalization of the Poisson theorem [19].

Theorem 1. Let $f_1, \ldots, f_q\,,\ q \geqslant 2p$ be such that the set of functions $(H_1, \ldots, H_{2p-1}, f_{i_1}, \ldots, f_{i_{2p-1}})$ is in involution (this implies, in particular, that the $f_i\,,\ i = 1, \ldots, q$, are constants of motion). Then the quantities $\{f_{i_1}, \ldots, f_{i_{2p}}\}$ are also constants of motion.

Definition 5. A function $c(x)$ will be called a *Casimir function* if $\{g_1, g_2, \ldots, g_{2p-1},\ c\} = 0$ for any set of functions $(g_1, g_2, \ldots, g_{2p-1})$. If one of the Hamiltonians $(H_1, \ldots, H_{2p-1})$ is a Casimir function, then the generalized dynamics defined by (24) is trivial.

As an example of these generalized Poisson structures we now show succinctly that any simple Lie algebra $\mathcal{G}$ of rank l provides a family of l generalized linear Poisson structures, and that each of them may be characterized by a cocycle in the Lie algebra cohomology.

We now turn to generalized Poisson structures on the duals of simple Lie algebras. Let $\mathcal{G}$ be the Lie algebra of a simple compact group G. In this case the de Rham cohomology ring on the group manifold G is the same as the Lie algebra cohomology ring $H_0^*(\mathcal{G}, \mathbb{R})$ for the trivial action. In its Chevalley–Eilenberg version the Lie algebra cocycles are represented by bi-invariant (i.e. left- and right-invariant and hence closed) forms on G [20] (see also [21], for example). For instance, if using the Killing metric k_{ij} we introduce the skew-symmetric order three tensor

$$\omega(e_i, e_j, e_k) := k([e_i, e_j], e_k) = C_{ij}^l k_{lk} = C_{ijk} \qquad e_i \in \mathcal{G}\ (i, j, k = 1, \ldots, r = \dim \mathcal{G}) \tag{25}$$

this defines by left translation a left-invariant (LI) form on G which is also right-invariant. The bi-invariance of ω then reads

$$\omega([e_l, e_i], e_j, e_k) + \omega(e_i, [e_l, e_j], e_k) + \omega(e_i, e_j, [e_l, e_k]) = 0 \tag{26}$$

where the e_i are now understood as LI vector fields on G obtained by left translation from the corresponding basis of $\mathcal{G} = T_e(G)$. Equation (26) (the Jacobi identity) thus implies a three cocycle condition on ω; as a result $H_0^3(\mathcal{G}, \mathbb{R}) \neq 0$ for any simple Lie algebra as is well known. In terms of the standard Poisson structure, this means that the linear structure defined by (12) is associated with a non-trivial three-cocycle on $\mathcal{G}$ and that $[\Lambda^{(2)}, \Lambda^{(2)}] = 0$ (equation (13)) is precisely the cocycle condition. This indicates that the generalized linear Poisson structures on $\mathcal{G}^*$ may be found by looking for higher order cocycles.

The cohomology ring of any simple Lie algebra of rank l is a free ring generated by the l (primitive) forms on G of odd order $(2m-1)$. These forms are associated with the l primitive symmetric invariant tensors $k_{i_1 \ldots i_m}$ of order m which may be defined on $\mathcal{G}$ and of which the Killing tensor $k_{i_1 i_2}$ is just the first example. For the A_l series $(su(l+1))$, for instance, these forms have order $3, 5, \ldots, (2l+1)$; other orders (but always including 3) appear for the different simple algebras (see, e.g., [21]). As a result, it is possible to associate a $(2m-2)$ skew-symmetric contravariant primitive tensor field linear in x_j with each symmetric invariant polynomial $k_{i_1 \ldots i_m}$ of order m. The case $m = 2$ leads to the $\Lambda^{(2)}$ of (12), (17). We shall not describe the theory in detail, but rather limit ourselves to illustrating the main theorem below with an example.

Theorem 2. Let $\mathcal{G}$ be a simple compact algebra, and let $k_{i_1 \ldots i_m}$ be a primitive invariant symmetric polynomial of order m. Then, the tensor $\omega_{\rho l_2 \ldots l_{2m-2} \sigma}$

$$\begin{aligned} \omega_{\rho l_2 \ldots l_{2m-2}\sigma} &:= \epsilon^{j_2 \ldots j_{2m-2}}_{l_2 \ldots l_{2m-2}} \tilde{\omega}_{\rho j_2 \ldots j_{2m-2}\sigma} \\ \tilde{\omega}_{\rho j_2 \ldots j_{2m-2}\sigma} &:= k_{i_1 \ldots i_{m-1}\sigma} C^{i_1}_{\rho j_2} \ldots C^{i_{m-1}}_{j_{2m-3} j_{2m-2}} \end{aligned} \tag{27}$$

is completely skew-symmetric, defines a Lie algebra cocycle† on $\mathcal{G}$ and

$$\Lambda^{(2m-2)} = \frac{1}{(2m-2)!} \omega_{l_1 \ldots l_{2m-2}}{}^{\sigma} x_\sigma \partial^{l_1} \wedge \ldots \wedge \partial^{l_{2m-2}} \tag{28}$$

defines a generalized Poisson structure on $\mathcal{G}$.

Proof. The theorem is proved using that the SNB $[\Lambda^{(2m-2)}, \Lambda^{(2m-2)}]$ is zero due to the cocycle condition satisfied by $\omega_{\rho l_2 \ldots l_{2m-2}\sigma}$. In particular

$$\{x_{i_1}, x_{i_2}, \ldots, x_{i_{2m-2}}\} = \omega_{i_1 \ldots i_{2m-2}}{}^{\sigma} x_\sigma \tag{29}$$

where the $\omega_{i_1 \ldots i_{2m-2}}{}^{\sigma}$ are the 'structure constants' defining the $(2m-1)$ cocycle and hence the generalized PS. In fact, it may be shown that different $\Lambda^{(2m-2)}$, $\Lambda^{(2m'-2)}$ tensors also commute with respect to the SNB and that they generate a free ring.

Note. The requirement of compactness is introduced to have a definite Killing–Cartan metric which then may be taken as the unit matrix; this allows us to identify upper and lower indices.

Example (Generalized PS on su(3)).* Let $\mathcal{G} = su(3)$. Besides the Killing metric (which leads to the standard linear PS on the dual space $su(3)^*$), $su(3)$ admits another symmetric ad-invariant polynomial which may be expressed as $\mathrm{Tr}(\lambda_i\{\lambda_j, \lambda_k\}) = 4d_{ijk}$ (the d_{ijk} are the constants appearing in the anticommutator of the Gell-Mann λ_i matrices, $\{\lambda_i, \lambda_j\} = \frac{4}{3}\delta_{ij} 1_3 + 2d_{ijk}\lambda_k$). Then, the new Poisson structure is defined by

$$\Lambda^{(4)} = \frac{1}{4!} \omega_{i_1 i_2 i_3 i_4}{}^{\sigma} x_\sigma \frac{\partial}{\partial x_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{i_4}} \qquad \omega_{\rho i_2 i_3 i_4 \sigma} := \frac{1}{2} \epsilon^{j_2 j_3 j_4}_{i_2 i_3 i_4} d_{k_1 k_2 \sigma} C^{k_1}_{\rho j_2} C^{k_2}_{j_3 j_4}\,. \tag{30}$$

In fact, the $\omega_{\rho j_2 j_3 j_4 \sigma}$ in (30) is what appears in the '4-commutators'

$$[T_{j_1}, T_{j_2}, T_{j_3}, T_{j_4}] = \omega_{j_1 j_2 j_3 j_4}{}^{\sigma} T_\sigma \qquad (T_i = \tfrac{1}{2}\lambda_i)$$

† The origin of (27) is easy to understand since given a symmetric invariant polynomial $k_{i_1 \ldots i_m}$ on $\mathcal{G}$, the associated skew-symmetric multilinear tensor $\omega_{i_1 \ldots i_{2m-1}}$ is given by

$$\omega(e_{i_1}, \ldots, e_{i_{2m-1}}) = \sum_{s \in S_{(2m-1)}} \pi(s) k([e_{s(i_1)}, e_{s(i_2)}], [e_{s(i_3)}, e_{s(i_4)}], \ldots, [e_{s(i_{2m-3})}, e_{s(i_{2m-2})}], e_{s(i_{2m-1})})$$

where $\pi(s)$ is the parity sign of the permutation $s \in S_{(2m-1)}$.

which are given by the sum $\sum_{s\in S_4}\pi(s)T_{s(j_1)}T_{s(j_2)}T_{s(j_3)}T_{s(j_4)}$ of the $4! = 24$ products of four T's, each one with the sign dictated by the parity $\pi(s)$ of the permutation $s \in S_4$ and which give, as the Lie algebra commutator does, an element of $\mathcal{G}$ in the right-hand side. It now is not difficult to check, using the symmetry of the d's and the properties of the structure constants (including the Jacobi identity), that $[\Lambda^{(4)}, \Lambda^{(4)}] = 0$. Thus, all properties of definition 2 are fulfilled and $\Lambda^{(4)}$ defines a GPB. We refer the reader to [18] for further details concerning the mathematical structure of the GPB and the contents of the associated generalized dynamics and its quantization. We shall conclude here by saying that this analysis could be extended to Lie superalgebras and super-Poisson structures.

This research has been partially supported by the CICYT (Spain) under grant AEN93-187. AP wishes to thank the Vicerectorate of research of Valencia University for making his stay in Valencia possible; JCPB wishes to acknowledge a FPI grant from the Spanish Ministry of Education and Science and the CSIC.

References

[1] Nambu Y 1973 Generalized Hamiltonian dynamics *Phys. Rev.* D **7** 2405–12
[2] Bayen F and Flato M 1975 Remarks concerning Nambu's generalized mechanics *Phys. Rev.* D **11** 3049–53
[3] Mukunda N and Sudarshan E. 1976 Relation between Nambu and Hamiltonian mechanics *Phys. Rev.* D **13** 2846–50
[4] Hirayama H 1977 Realization of Nambu mechanics: A particle interacting with an $SU(2)$ monopole *Phys. Rev.* D **16** 530–2
[5] Takhtajan L 1994 On foundations of the generalized Nambu mechanics *Commun. Math. Phys.* **160** 295–315
[6] Chatterjee R 1995 Dynamical symmetries and Nambu mechanics *Stony Brook preprint*
[7] Lichnerowicz A 1977 Les variétés de Poisson et leurs algèbres de Lie associées *J. Diff. Geom.* **12** 253–300
[8] Weinstein A 1983 The local structure of Poisson manifolds *J. Diff. Geom.* **18** 523–57
[9] Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D 1978 Deformation theory and quantization *Ann. Phys.* **111** 61–151
[10] Lie S 1874/75 Begründung einer Invariantentheorie der Berührungs Transformationen *Math. Ann.* **8** 214–303
[11] Lie S and Engel F 1888 *Theorie der Transformationsgruppen I–III* (Leipzig: Teubner) (reprinted 1970 (Chelsea))
[12] Grabowski J, Marmo G and Perelomov A M 1993 Poisson structures: towards a classification *Mod. Phys. Lett.* **18A** 1719–33
[13] Cariñena J, Ibort A, Marmo G and Perelomov A M 1994 On the geometry of Lie algebras and Poisson tensors *J. Phys. A: Math. Gen.* **27** 7425–49
[14] Alekseevsky D V and Perelomov A M 1995 Poisson brackets on Lie algebras *Preprint* ESI 247
[15] Tulczyjev W M 1974 Poisson brackets and canonical manifolds *Bull. Acad. Pol. Sci. (Math. Astron.)* **22** 931–4
[16] Schouten J A 1940 Ueber Differentialkomitanten zweir kontravarianter Gröszen *Proc. Kon. Ned. Akad. Wet.* **43** 449–52
[17] Nijenhuis A 1955 Jacobi-type identities for bilinear differential concomitants of certain tensor fields *Indag. Math.* **17** 390–403
[18] de Azcárraga J A, Perelomov A M and Pérez Bueno J C 1996 in preparation
[19] Poisson S 1809 Mémoire sur la variation des constantes arbitraires dans les questions de méchanique *J. École Polytec.* **8** 266–344
[20] Chevalley C and Eilenberg S 1948 Cohomology theory of Lie groups and Lie algebras *Trans. Am. Math. Soc.* **63** 85–124
[21] de Azcárraga J A and Izquierdo J M 1995 *Lie Algebras, Lie Groups, Cohomology and Some Applications in Physics* (Cambridge: Cambridge University Press)